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1990 J. Phys. A: Math. Gen. 23 1985

(http://iopscience.iop.org/0305-4470/23/11/022)

J. Phys. A: Math. Gen. 23 (1990) 1985-1991. Printed in the UK

A note on non-integrable phases and coherent states

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Received 28 July 1989

Abstract. The relationship between non-integrable phases and generalised coherent states is discussed.

1. General formalism

It has recently become more and more evident that the notion of non-integrable phase factor (the Berry phase (BP) in the case of adiabatic process (Berry 1984) and the Aharonov-Anandan phase (AAP) in the case of general cyclic processes (Aharonov and Anandan 1987)) provides a very useful tool in understanding many quantum mechanical phenomena.

In many applications one is often faced with the problem of calculating the BP or AAP for the family of Hamiltonians constructed in the following way. The space of states carries a unitary representation $\{U(g)|g\in \mathrm{G}\}$ of some Lie group G. Given a parameter space M, the Hamiltonian $H(\xi)$ belongs to the Lie algebra of G for any $\xi\in M$ (the special case of the family of unitary equivalent Hamiltonians, after some redefinition of timescale, has been recently considered by Giavarini and Onofri (1989)). Suppose we have a path $\xi=\xi(t)$ and consider the Schrödinger equation

$$\mathrm{i}\frac{\mathrm{d}\psi(t)}{\mathrm{d}t}=H(\xi(t))\psi(t) \qquad \psi(t=0)=\psi_0. \tag{1}$$

Now, under the above assumptions, the evolution operator

$$U(t)=T\exp\left(-\mathrm{i}\int_0^t \mathrm{d}\tau\, H(\xi(\tau))\right)\equiv U(g(t))$$

belongs to the representation of G. Let $\mathrm{S}\subset \mathrm{G}$ be the subgroup consisting of all elements $g\in \mathrm{G}$ such that

$$U(g)\psi_0=\mathrm{e}^{\mathrm{i}\alpha(g)}\psi_0. \tag{2}$$

Let ζ parametrise the coset space G/S. Then for any $g\in \mathrm{G}$, $g=\zeta\cdot h$, $\zeta\in \mathrm{G/S}$, $h\in \mathrm{S}$. Therefore we may write

$$\psi(t)=U(t)\psi_0=\mathrm{e}^{\mathrm{i}\phi(t)}U(\zeta(t))\psi_0 \qquad \phi(t)=\alpha(h(t))$$
$$g(t)=\zeta(t)h(t). \tag{3}$$

|| Supported by CPBP 01.03.

But $\psi(\zeta) \equiv U(\zeta)\psi_0$ is nothing but a generalised coherent state (Perelomov 1987, Giavarini and Onofri 1989). Consequently the solution to the Schrödinger equation (1) is to be looked for in the form

$$\psi(t) = e^{i\phi(t)}\psi(\zeta(t)). \tag{4}$$

Let us write

$$H(\xi) = a^{\lambda}(\xi)T_{\lambda} + a^{i}(\xi)T_{i} \qquad \xi \in M$$

where T_λ and T_i are the generators corresponding to S and G/S, respectively. We introduce the standard Cartan forms $\omega_i^{\lambda}(\zeta)$ and $\eta_i^{j}(\zeta)$ by

$$iU^{+}(\zeta)\partial_i U(\zeta) = \omega_i^{\lambda}(\zeta)T_{\lambda} + \eta_i^{j}(\zeta)T_j.$$

Substituting the expression (3) (or (4)) into the Schrödinger equation (1) we obtain

$$\begin{aligned}[-\dot{\phi} + \dot{\zeta}^{i}(\omega_i^{\lambda}(\zeta)T_{\lambda} + \eta_i^{j}(\zeta)T_j)]\psi_0 \\ = [D_{\lambda'}^{\lambda}(\zeta)a^{\lambda'}(\xi) + D_i^{\lambda}(\zeta)a^{i}(\xi)]T_{\lambda}\psi_0 + [D_{\lambda}^{i}(\zeta)a^{\lambda}(\xi) + D_j^{i}(\zeta)a^{j}(\xi)]T_i\psi_0\end{aligned} \tag{5}$$

where $D(\ldots)$ is the adjoint representation of G. Let us note that

(i) $T_\lambda\psi_0 = \tau_\lambda\psi_0 \qquad \tau_\lambda \in \mathbb{R}$

(ii) the vectors ψ_0, $T_i\psi_0$ are linearly independent over $\mathbb{R}$.

These facts follow immediately from the definition of the subgroup S. Using (i) and (ii) we obtain from (5)

$$\dot{\zeta}^{i}\eta_i^{j}(\zeta) = D_{\lambda}^{j}(\zeta)a^{\lambda}(\xi) + D_i^{j}(\zeta)a^{i}(\xi) \tag{6a}$$

$$\dot{\phi} - \dot{\zeta}^{i}\omega_i^{\lambda}(\zeta)\tau_{\lambda} = -D_{\lambda'}^{\lambda}(\zeta)a^{\lambda'}(\xi)\tau_{\lambda} - D_i^{\lambda}(\zeta)a^{i}(\xi)\tau_{\lambda}. \tag{6b}$$

These equations replace the Schrödinger one. They may be described as follows: the first set, (6*a*), describes some dynamics on G/S; we have to find the functions $\zeta = \zeta(t)$ provided the functions $\xi = \xi(t)$ are given. After solving them we may calculate the phase $\phi(t)$ from (6*b*).

If it happens that for some function $\xi = \xi(t)$, $0 \leqslant t \leqslant T$, there exists the solution $\zeta(t)$ such that $\zeta(0) = \zeta(T)$, then the quantum system under consideration performs the cyclic motion in the sense of Aharonov and Anandan; $\phi(T)$ is the full phase it develops. After subtracting the dynamical phase we obtain the AAP. More precisely, the AAP is defined as follows: if $\psi(T) = e^{i\phi}\psi(0)$ then

$$\alpha = i\int_0^T dt(\psi(t), \dot{\psi}(t)) + \phi.$$

Inserting here $\psi(t)$ as given by (3) we get (using $\phi \equiv \phi(T) - \phi(0)$)

$$\alpha = \oint d\zeta^{i}[\omega_i^{\lambda}(\zeta)\tau_{\lambda} + \eta_i^{j}(\zeta)(\psi_0, T_j\psi_0)]. \tag{7}$$

Note that the AAP is defined entirely in terms of Cartan forms on G/S (cf Giler *et al* 1989).

Assume now that the Hamiltonian $H(\xi)$ itself performs a cyclic motion, $\xi(0) = \xi(T)$. To obtain the adiabatic approximation we note that the condition that $U(\zeta(t))\psi_0$ is an eigenvector of $H(\xi)$ is equivalent to the equality

$$D_{\lambda}^{i}(\zeta)a^{\lambda}(\xi) + D_j^{i}(\zeta)a^{j}(\xi) = 0. \tag{8}$$

Then the eigenvalue condition $H(\xi)U(\zeta)\psi_0 = EU(\zeta)\psi_0$ gives

$$E = (D_{\lambda'}^{\lambda}(\zeta)a^{\lambda'}(\xi) + D_i^{\lambda}(\zeta)a^{i}(\xi))\tau_{\lambda}.$$

Taking a scalar product of both sides of (5) with ψ_0 we arrive at the formula for BP identical to that for AAP, equation (7) (cf Brihaye and Kosiński 1989), but supplemented with (8) instead of (6*a*).

2. Some examples

We shall consider some examples of non-integrable phases, mostly already discussed in the literature. As a first example consider the harmonic oscillator under the influence of external force.

Let us consider the group G generated by the elements $N=\frac{1}{2}(p^2+q^2)$, p, q, I. We start with some eigenvector $\tilde{\psi}_n$ of N. The equations (6) read $(\psi(\zeta)\equiv\exp \mathrm{i}(Pq-Qp)\psi_n)$

$$\dot{Q}=P$$
$$\dot{P}=-Q+f(t)$$
$$\dot{\phi}+\tfrac{1}{2}Q\dot{P}-\tfrac{1}{2}P\dot{Q}=f(t)Q-\frac{Q^2+P^2}{2}-(n+\tfrac{1}{2}).$$

Note that $(\tilde{\psi}_n, p\tilde{\psi}_n)=(\tilde{\psi}_n, q\tilde{\psi}_n)=0$; therefore we need only the Cartan form corresponding to the stability subgroup generated by I and N. It reads (see the appendix for the calculation of Cartan forms)

$$\tfrac{1}{2}(P\,\mathrm{d}Q-Q\,\mathrm{d}P)\cdot I.$$

The Berry phase is trivial here because the parameter space is one dimensional. However, for some choices of $f(t)$ we may have the solutions with $Q(0)=Q(T)=0$, $P(0)=P(T)=0$. Then the AAP is given by†

$$\alpha=\frac{1}{2}\oint(P\,\mathrm{d}Q-Q\,\mathrm{d}P)=\oint P\,\mathrm{d}Q=-\int \mathrm{d}Q\,\mathrm{d}P.$$

As a second example let us consider the 'classical' case of the spin in the external magnetic field. The relevant group is SU(2) while the stability subgroup is U(1). The coherent states read

$$|\xi\rangle=\mathrm{e}^{\xi J_+-\bar{\xi}J_-}|j,m\rangle.$$

Again we need only the Cartan form related to the subgroup, because $\langle j,m|J_\pm|j,m\rangle=0$. It reads (see the appendix)

$$\omega=\frac{\mathrm{i}(\xi\,\mathrm{d}\bar{\xi}-\bar{\xi}\,\mathrm{d}\xi)}{2|\xi|^2}(1-\cos 2|\xi|).$$

We arrive therefore at the following result for AAP or BP:

$$\alpha=\mathrm{i}m\oint\frac{(1-\cos 2|\xi|)}{2|\xi|^2}(\xi\,\mathrm{d}\bar{\xi}-\bar{\xi}\,\mathrm{d}\xi).$$

But there exists the following relation between ξ and the unit vector $\boldsymbol{n}=(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$: $\xi=-\frac{1}{2}\theta\,\mathrm{e}^{-\mathrm{i}\varphi}$ (Perelomov 1986); with this relation we can rederive the result of Berry.

† Note that this result in fact does not depend on the choice of ψ_0 because other Cartan forms are exact differentials.

The third example is provided by the harmonic oscillator with time-dependent mass and frequency (in fact, by a suitable change of time variable this is reduced to the case of constant mass). The Schrödinger equation

$$\mathrm{i}\frac{\mathrm{d}\psi(t)}{\mathrm{d}t} = H(t)\psi(t) \qquad H(t) \equiv \tfrac{1}{2}(P^2 + \omega^2(t)q^2)$$

may be written as (Perelomov 1986)

$$\mathrm{i}\dot{\psi}(t) = \boldsymbol{\Omega}(t)\boldsymbol{K}\psi(t) \qquad \boldsymbol{\Omega}\cdot\boldsymbol{K} \equiv \Omega_3 K_3 - \Omega_1 K_1 - \Omega_2 K_2$$

where

$$K_{1,3} = \frac{1}{4}\left(\frac{p^2}{\omega(0)} \mp \omega(0)q^2\right) \qquad K_2 = \frac{1}{4}(pq + qp)$$

$$\Omega_{1,3} = \omega(0)\left(\frac{\omega^2(t)}{\omega^2(0)} \mp 1\right) \qquad \Omega_2 = 0.$$

It is easy to check that the operators K_i, $i = 1, 2, 3$, form the Lie algebra of SU(1, 1); in terms of the creation and annihilation operators at $t = 0$ they read ($K_\pm = \mathrm{i}(K_\pm \mathrm{i} K_2)$:

$$K_+ = \frac{(a^+)^2}{2} \qquad K_- = \frac{a^2}{2} \qquad K_3 = \tfrac{1}{4}(aa^+ + a^+a).$$

Noting that $H(0) = 2\omega(0)K_3$ we conclude that if ψ_0 is an eigenvector of $H(0)$, the subgroup S is the U(1) group generated by K_3. Therefore the coset space parametrising the coherent space is

$$\mathrm{G/S} = \{\boldsymbol{n} : n_3^2 - n_1^2 - n_2^2 = 1, n_3 > 0\}.$$

The coherent states read

$$|\xi\rangle = \mathrm{e}^{(\xi K_+ - \bar{\xi}K_-)}\psi_0$$

where $\xi = -\frac{1}{2}\tau\,\mathrm{e}^{-\mathrm{i}\phi}$, $\boldsymbol{n} = (\sinh\tau\cos\varphi, \sinh\tau\sin\varphi, \cosh\tau)$. To calculate BP or AAP note that $(\psi_0, K_\pm\psi_0) = 0$ for any eigenstate of $H(0)$. The Cartan form corresponding to the subgroup reads

$$\omega = \frac{-\mathrm{i}(\xi\,\mathrm{d}\bar{\xi} - \bar{\xi}\,\mathrm{d}\xi)}{2|\xi|^2}(1 - \cosh 2|\xi|).$$

The non-integrable phase reads

$$\alpha = -\mathrm{i}E(0)\oint_C \frac{(\xi\,\mathrm{d}\bar{\xi} - \bar{\xi}\,\mathrm{d}\xi)}{2|\xi|^2}(1 - \cosh 2|\xi|).$$

Again it is proportional to the area† of part of a hyperboloid $n_3^2 - n_1^2 - n_2^2 = 1$ encircled by the loop C. Actually, in our case only the AAP is non-trivial. However, we may allow $\Omega_2 \neq 0$, i.e. consider the generalised oscillator with the term $pq + qp$ added. Nothing will then change in our conclusions.

Let us now pass to the more general case of the harmonic oscillator with time-dependent frequency under the influence of time-dependent force

$$H(t) = \tfrac{1}{2}(p^2 + \omega^2(t)q^2) - f(t)q.$$

† Of course, we mean here the invariant measure on the Lobachevsky plane.

With the help of generators K_i and the creation and annihilation operators at $t=0$ we may write

$$H(t)=\boldsymbol{\Omega}(t)\boldsymbol{K}-\frac{f(t)}{\sqrt{2\omega(0)}}(a+a^+).$$

The operators K_i, a, a^+ and I form the Lie algebra. Starting with the eigenstate of the unperturbed Hamiltonian at $t=0$ we see that the stability group is generated by K_3 and I. The coherent states read (for a choice of parametrisation here see the appendix)

$$|\xi, z\rangle = \mathrm{e}^{\xi K_+ - \bar{\xi} K_-}\, \mathrm{e}^{za^+ - \bar{z}a}\, |\psi_0\rangle.$$

We check that $(\psi_0, K_{\pm\_}\psi_0)=(\psi_0, a\psi_0)=(\psi_0, a^+\psi_0)=0$; the Cartan form corresponding to the subgroup is

$$\omega K_3+\frac{\mathrm{i}}{2}(\bar{z}\,\mathrm{d}z-z\,\mathrm{d}\bar{z})I+\tfrac{1}{2}(z^2\eta_-+\bar{z}^2\eta_++|z|^2\omega)\cdot I$$

where $\eta_\pm$ and ω correspond to the subgroup SU(1, 1) (see the appendix). With the above form we can calculate the non-integrable phases.

We conclude with the following remark. As it has been shown in the book of Perelomov (1986) the SU(1, 1) group is the dynamical symmetry group also for the singular oscillator

$$H=\tfrac{1}{2}(p^2+\omega^2(t)q^2)+\frac{a^2}{q^2}.$$

This allows us to calculate the AAP in the same way as above.

3. Final remarks

As it has already been noticed by Giavarini and Onofri (1989) the coherent states provide a convenient tool to analyse the non-integrable phases for certain class of Hamiltonians. This class is quite large; it contains Hamiltonians expressible in terms of generators of some Lie algebra. We have presented some rather standard examples. For other physically interesting ones see the book by Perelomov (1986).

We have expressed the non-integrable phase in terms of a loop integral over the Cartan forms. This makes its geometrical origin transparent. Moreover, the existence of a systematic algorithm for calculating the Cartan forms makes it possible to determine effectively the Aharonov–Anandan and Berry phases.

Appendix

We show how to calculate the Cartan forms. It is a very simple task for nilpotent groups while for others the straightforward use of the Hausdorff formula seems to be ineffective. We use the method of Volkov and Pervushin (1978). The Lie algebra of the group under consideration reads

$$\begin{aligned}
&[T_\alpha, T_\beta]=\mathrm{i}C^{\gamma}_{\alpha\beta}T_\gamma\\
&[T_\alpha, T_i]=\mathrm{i}C^{k}_{\alpha i}T_k+\mathrm{i}C^{\beta}_{\alpha i}T_\beta\\
&[T_i, T_j]=\mathrm{i}C^{k}_{ij}T_k+\mathrm{i}C^{\alpha}_{ij}T_\alpha.
\end{aligned}\tag{A1}$$

The Cartan forms are defined as follows:

$$\Omega \equiv \mathrm{i}\,\mathrm{e}^{-\mathrm{i}\xi^k T_k}\,\mathrm{d}\,\mathrm{e}^{\mathrm{i}\xi^k T_k} \equiv \omega^\lambda T_\lambda + \eta^k T_k.$$

To calculate Ω we introduce the additional parameter t

$$\Omega(t) \equiv \mathrm{i}\,\mathrm{e}^{-\mathrm{i}t\xi^k T_k}\,\mathrm{d}\,\mathrm{e}^{\mathrm{i}t\xi^k T_k} \equiv \omega^\lambda(t) T_\lambda + \eta^k(t) T_k.$$

Differentiating with respect to t and using (A1) one gets

$$\dot\Omega(t) = \mathrm{i}[\Omega(t), \xi^k T_k] - \mathrm{d}\xi^k T_k$$

or

$$\begin{aligned} \dot\omega^\lambda &= -C^\lambda_{ik}\eta^i\xi^k - C^\lambda_{\alpha k}\omega^\alpha\xi^k \\ \dot\eta^i &= -\mathrm{d}\xi^i - C^i_{\lambda k}\omega^\lambda\xi^k - C^i_{jk}\eta^j\xi^k. \end{aligned} \tag{A2}$$

Solving (A2) with the boundary condition $\omega^\lambda(t=0) = \eta^i(t=0) = 0$ and taking $t=1$ we obtain the Cartan forms.

The SU(2) group

We define

$$\mathrm{i}\,\mathrm{e}^{-(\xi J_+ - \bar\xi J_-)}\,\mathrm{d}\,\mathrm{e}^{(\xi J_+ - \bar\xi J_-)} \equiv \eta_+ J_+ + \eta_- J_- + \omega J_3.$$

The equations (A2) read

$$\begin{aligned} \dot\omega &= -2(\eta_+\bar\xi + \eta_-\xi) \\ \dot\eta_+ &= \mathrm{i}\,\mathrm{d}\xi + \omega\xi \\ \dot\eta_- &= -\mathrm{i}\,\mathrm{d}\bar\xi + \omega\bar\xi. \end{aligned}$$

The solution to them is

$$\begin{aligned} \omega &= \frac{\mathrm{i}(\xi\,\mathrm{d}\bar\xi - \bar\xi\,\mathrm{d}\xi)}{2|\xi|^2}(1 - \cos 2|\xi|) \\ \eta_+ &= \mathrm{i}\,\mathrm{d}\xi + \frac{\mathrm{i}\xi(\xi\,\mathrm{d}\bar\xi - \bar\xi\,\mathrm{d}\xi)}{2|\xi|^2}\left(1 - \frac{\sin 2|\xi|}{2|\xi|}\right) \\ \eta_- &= \overline{\eta_+}. \end{aligned}$$

The SU(1, 1) group

With the same parametrisation

$$\mathrm{i}\,\mathrm{e}^{-(\xi K_+ - \bar\xi K_-)}\,\mathrm{d}\,\mathrm{e}^{(\xi K_+ - \bar\xi K_-)} \equiv \eta_+ K_+ + \eta_- K_- + \omega K_3$$

we get

$$\begin{aligned} \dot\omega &= 2(\bar\xi\eta_+ + \xi\eta_-) \\ \dot\eta_+ &= \mathrm{i}\,\mathrm{d}\xi + \omega\xi \\ \dot\eta_- &= -\mathrm{i}\,\mathrm{d}\bar\xi + \omega\bar\xi \end{aligned}$$

and

$$\begin{aligned} \omega &= \frac{-\mathrm{i}(\xi\,\mathrm{d}\bar\xi - \bar\xi\,\mathrm{d}\xi)}{2|\xi|^2}(1 - \cosh 2|\xi|) \\ \eta_+ &= \mathrm{i}\,\mathrm{d}\xi - \frac{\mathrm{i}\xi(\xi\,\mathrm{d}\bar\xi - \bar\xi\,\mathrm{d}\xi)}{2|\xi|^2}\left(1 - \frac{\sin 2|\xi|}{2|\xi|}\right) \\ \eta_- &= \overline{\eta_+}. \end{aligned}$$

The extended Heisenberg group. Consider the group generated by $N=a^+a+\frac{1}{2}$, a, a^+ and I and the one generated by I and N as a subgroup. The Cartan form reads

$$\Omega=\mathrm{i}\,\mathrm{e}^{-(za^+-\bar{z}a)}\,\mathrm{d}\,\mathrm{e}^{(za^+-\bar{z}a)}\equiv\omega_N N+\omega I+\eta_+a^++\eta_-a.$$

Our method gives

$$\Omega=\frac{\mathrm{i}}{2}(\bar{z}\,\mathrm{d}z-z\,\mathrm{d}\bar{z})I+\mathrm{i}\,\mathrm{d}z\,a^+-\mathrm{i}\,\mathrm{d}\bar{z}\,a.$$

Finally let us note that in some cases it is convenient to choose another parametrisation. Assume that the algebra under consideration is the sum of two subalgebras, $\mathrm{L}=\mathrm{T}+\mathrm{M}$. If we choose the parametrisation (taking from both subalgebras only the generators corresponding to the coset space)

$$\mathrm{e}^{\mathrm{i}\xi^k\mathrm{T}_k}\,\mathrm{e}^{\mathrm{i}\lambda^a\mathrm{M}_a}$$

then

$$\begin{aligned}\Omega^{\mathrm{T}+\mathrm{M}}&\equiv\mathrm{e}^{-\mathrm{i}\lambda^a\mathrm{M}_a}\,\mathrm{e}^{-\mathrm{i}\xi^k\mathrm{T}_k}\,\mathrm{d}(\mathrm{e}^{\mathrm{i}\xi^k\mathrm{T}_k}\,\mathrm{e}^{\mathrm{i}\lambda^a\mathrm{M}_a})\\&=\Omega^{\mathrm{M}}+\mathrm{e}^{-\mathrm{i}\lambda^a\mathrm{M}_a}\,\Omega^{\mathrm{T}}\,\mathrm{e}^{\mathrm{i}\lambda^a\mathrm{M}_a}\end{aligned}$$

which is often simpler to calculate. We use this method to calculate the Cartan forms for $\mathrm{T}=\{K_1, K_2, K_3\}$ and $\mathrm{M}=\{a, a^+, I\}$.

References

Aharonov Y and Anandan J 1987 *Phys. Rev. Lett.* **58** 159
Berry M V 1984 *Proc. R. Soc.* A **392** 45
Brihaye Y and Kosiński P 1989 *Acta Phys. Polon.* B **20** 223
Giavarini G and Onofri E 1989 *J. Math. Phys.* **30** 659
Giler S, Kosiński P and Szymanowski L 1989 *Int. J. Mod. Phys.* A **4** 1453
Perelomov A 1986 *Generalised Coherent States* (Berlin: Springer)
Volkov M K and Pervushin V N 1978 *Essentially Nonlinear Quantum Theories, Dynamical Symmetries and Meson Physics* (Moscow: Atomizdat) (in Russian)